ABSTRACT
In this paper, a nonlinear controller is designed for a wheeled mobile robot with three omnidirectional wheels. The omnidirectional wheel has an advantage of universal moving directions on a flat surface without changing its pose. The dynamics of the omnidirectional wheel includes rolling and slipping. Three configurations of mobile robots and an input-additive perturbation are considered. Furthermore, the maneuverability of the omnidirectional mobile robot is analyzed with the vehicle’s controllability that provides a measure on the omnidirectional mobile robots ability to track a trajectory. In controller design, a computed torque control law is applied to counterbalance the nonlinear terms of the system. Then a sliding mode controller is developed for trajectory tracking control. The effectiveness of the proposed control strategy is demonstrated by computer simulations.

KEY WORDS
Sliding mode control, Mobile robots, Omnidirectional wheels, Tracking control

1. Introduction
Mobile robots have attracted much attention of many researchers because of their flexible and versatile functions in many applications. The applications for mobile robots involve working at the places where are not suitable for humans or cannot be easily achieved by humans. The omnidirectional motion is enabled via non-conventional wheels. Fig. 1 shows the commercial omnidirectional wheel used in our mobile robot. It has an advantage of moving in any arbitrary course without changing the direction of wheels.

The objective of our research is to develop a robust control approach to ensure fast and smooth motion of the omnidirectional mobile robot. Many nonlinear controllers have been proposed to solve these problems [14][15]. It was assumed that there were some kinds of dynamic controllers that could produce exactly the same velocity that was necessary for the kinematic controllers. In [8], a third PID controller received the desired final angular velocity and managed the wheels. In [16], a linear quadratic regulator controller was developed for a wheeled robot. Virtual simulations were applied to evaluate motion of a mobile robot in constrained plan with obstacles using multiple sensors [17]. In [14][18], robust tracking controllers using the sliding mode approach were proposed for nonholonomic wheeled mobile robot.

There have been many possible control methods can be used, whether the kinematic model of the controlled plant is complete or not, like the PID control method and the robust control methods. In this paper, a robust sliding mode controller that can achieve perfect tracking for the mobile robot in the presence of parametric uncertainties.
and external disturbances like slipping or traction forces is developed. The dynamic model of the omnidirectional mobile robot will be derived first. Then the computed torque control law will be designed to balance the nonlinear terms of the system. After that, a sliding mode controller dealing with the modeling uncertainty will be implemented to accomplish trajectory tracking control tasks. It will be shown that the proposed controller can effectively overcome uncertainties and external disturbances. The stability of the control system is also proved using the Lyapunov method.

The remainder of this paper is organized as follows: Section 2 describes the nonlinear mathematical model of an omnidirectional mobile robot system. In Section 3, the design of a sliding mode controller for a mobile robot is addressed. The global bounded property is also guaranteed using the Lyapunov method. In Section 4, experiments are performed to verify the effectiveness of the presented controller. Finally, some conclusions are given in Section 5.

2. Dynamic Modeling of the Omni-Directional Mobile Robot with Wheel Slip

The dynamic model of the omnidirectional mobile robot is a nonholonomic system, which includes a slipping function. The corresponding dynamic model can be derived by using the Newton’s law [7]. The base platform is supported by three omnidirectional wheels at an equal distance from the center of gravity for the robot. Therefore, the angles for two neighboring wheels are all 120 degrees, as show in Fig. 1. Symbols used in the dynamic model are listed in the nomenclature.

It is assumed that the wheels have no slipping in the direction of the traction force. There are two coordinate frames used in the robot modeling, i.e., the body frame and the world frame. By Newton's second law, we have

\[
\begin{align*}
\begin{bmatrix}
  m a_{xw} \\
  m a_{yw} \\
  I_x \phi 
\end{bmatrix} = 
\begin{bmatrix}
  F_x \\
  F_y \\
  M 
\end{bmatrix}
\end{align*}
\]

(1)

and

\[
\begin{align*}
\begin{bmatrix}
  a_{xw} \\
  a_{yw} \\
  \phi 
\end{bmatrix} = 
\begin{bmatrix}
  H F_x \\
  H F_y \\
  M 
\end{bmatrix}
\end{align*}
\]

(2)

where

\[
H = \begin{bmatrix}
\frac{1}{m} & 0 & 0 \\
0 & \frac{1}{m} & 0 \\
0 & 0 & \frac{1}{I_x}
\end{bmatrix}
\]

\[
F = [F_x, F_y]^T
\]

is the vector in the world coordinate applied to the center of gravity and \(M\) is the moment of inertia with respect to the center of gravity for the robot. Let \(\phi\) denote the angle between the world frame \((x_w, y_w)\) and body frame \((x_m, y_m)\) such that coordinate transformations from the world coordinate to the body coordinate can be expressed as follows:

\[
v_w = ^wS_m v_m
\]

(3)

\[
F = ^wS_m f
\]

(4)

where

\[
^wS_m = \begin{bmatrix}
\cos(\phi) & -\sin(\phi) \\
\sin(\phi) & \cos(\phi)
\end{bmatrix}
\]

(5)

Define \(v_m = [v_{x_m} v_{y_m}]^T\) to be the velocity vector of the center of gravity and \(f = [f_x f_y]^T\) as the force vector applied to the center of gravity. \(^wS_m\) stands for the coordinate transformation matrix from the world frame to the body frame. From Fig. 1, the dynamics of omnidirectional mobile robot can be described as
\[ f_x = \cos(0)f_1 - \sin(\delta)f_2 - \sin(\delta)f_3 \]
\[ f_y = \sin(0)f_1 + \cos(\delta)f_2 - \cos(\delta)f_3 \]
\[ M = Lf_1 + Lf_2 + Lf_3 \]

For the purpose of design convenience, the dynamic equation of (6) is partitioned into the following form:

\[
\begin{bmatrix}
  f_x \\
  f_y \\
  M
\end{bmatrix} = B_k \begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3
\end{bmatrix} \tag{7}
\]

where

\[
B_k = \begin{bmatrix}
  1 & -\sin(\delta) & -\sin(\delta) \\
  0 & \cos(\delta) & -\cos(\delta) \\
  L & L & L
\end{bmatrix} \tag{8}
\]

Incorporating equations (3) and (8) into (2) leads

\[
\begin{bmatrix}
  a_{xw} \\
  a_{yw} \\
  \phi
\end{bmatrix} = H^\top R_m B_k \begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3
\end{bmatrix} \tag{9}
\]

where \( ^\top R_m \) is the orthonormal rotation matrix giving the orientation of the moving frame with respect to the world frame, i.e.,

\[
^\top R_m = \begin{bmatrix}
  \cos(\phi) & -\sin(\phi) & 0 \\
  \sin(\phi) & \cos(\phi) & 0 \\
  0 & 0 & 1
\end{bmatrix} \tag{10}
\]

According to the wheel velocities of an omnidirectional mobile robot and wheel assembly analysis illustrated in Figs. 2 and 3, we can rewrite the kinematic equations:

\[
\begin{align*}
\|v_1\| &= R\dot{\theta}_1 = v_{sw} + L\dot{\phi} \\
\|v_2\| &= R\dot{\theta}_2 = -\sin\left(\frac{\pi}{6}\right)v_{sw} + \cos\left(\frac{\pi}{6}\right)v_{sw} + L\dot{\phi} \\
\|v_3\| &= R\dot{\theta}_3 = -\sin\left(\frac{\pi}{6}\right)v_{sw} - \cos\left(\frac{\pi}{6}\right)v_{sw} + L\dot{\phi}
\end{align*} \tag{11}
\]

The dynamic model of a DC motor system can be decomposed into an electrical and a mechanical subsystems, i.e.,

\[ u = L_a \frac{di}{dt} + R_i + k_a \omega_m \]

and

\[ J_m \frac{d\omega_m}{dt} = k_e i - c_a \omega_m - \tau \]

where \( u \) denotes the armature voltage, \( L_a \) and \( R_i \) are the inductance and the resistance of the armature, \( i \) is the armature current, \( k_e \) is the back emf constant, \( \omega_m \) is the
motor shaft speed, $J_m$ is the combined inertia of the motor, $k_i$ is the motor torque constant, $c_m$ is the viscous friction coefficient of the motor, and $\tau$ stands for the torque applied at the wheel contact.

By identifying the motor parameters, we can assume that $L_m$, $J_m$, and $c_m$ are very small, then the motor’s dynamical behavior can be simplified to

$$u = R_i i + k_i \omega_m$$
$$\tau = k_i i$$

(15)

The relationship between the control torque $\tau$ and the control voltage $u$ is given by

$$\tau = \frac{k_i}{R_a} u - \frac{k_i^2}{R_a} \omega_m$$

where $\omega_m = \omega$

(16)

Form Fig. 4, the control torque $\tau$ and the output force $f$ has the following relationship:

$$f = \frac{n}{R} \tau$$

(17)

where $n$ is the gear ratio and can be set for 1 in this specific case. By combining equations (15), (16), and (17), we can rewrite the dynamic equation to

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} n k_b \\ \frac{n^2 h_b^2}{R R_a} \omega_2 \\ u_3 \end{bmatrix}$$

(18)

Applying equations (9), (14), and (18) with the vector notation, the dynamic equations in the world frame for those three identical motors can be formulated as:

$$\begin{align*}
\dot{x}_u &= a_1 \dot{x}_u + 2 h_1 \cos(\phi) u_i + \left( -h_1 \cos(\phi) - \sqrt{3} h_1 \sin(\phi) \right) u_2 \\
\dot{y}_u &= a_2 \dot{y}_u + 2 h_2 \sin(\phi) u_i + \left( -h_2 \sin(\phi) + \sqrt{3} h_1 \cos(\phi) \right) u_2 \\
\dot{\phi} &= a_3 \phi + b_3 u_1 + b_2 u_2 + b_4 u_3
\end{align*}$$

(19)

where

$$\begin{align*}
a_1 &= -3 n^2 k_s^2 \left( 2 R^2 m R_s \right), \\
a_2 &= -3 n^2 k_s^2 L \left( R^2 I R_s \right) \\
b_1 &= n k_s \left( 2 R m R_s \right), \\
b_2 &= n k_s L \left( R I R_s \right)
\end{align*}$$

The Dahl model introduced in [19] was developed for the purpose of simulating control systems with friction. The friction model where the adhesion coefficient between the wheels of the mobile robot and a flat surface is a function of the wheel slip is applied in this study. General models of friction always consist of different components, which take care of certain aspects of the friction force. Let $\beta$ be the displacement, $F$ be the friction force developed at the wheel, and $F_c$ be the Coulomb friction force. The friction model proposed by Dahl has the following form:

$$\frac{dF}{d\beta} = \sigma \left( 1 - \frac{F}{F_c} \text{sgn}\left( \beta \right) \right)^i$$

(20)

where $\sigma$ is the contact stiffness and $i$ governs the transition rate of $F$ in order to achieve a better experimental match. The case of $i = 1$ is most commonly used.

Then, after a number of mathematical manipulations the dynamic equations reduce to

$$\begin{align*}
\dot{P}_u &= A_u P_u + B_u(\phi) U_c + D F_f \\
\end{align*}$$

(21)

where

$$\begin{align*}
P_u &= \begin{bmatrix} x_u & y_u & \phi \end{bmatrix}^T \\
U_c &= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T \\
A_u &= \begin{bmatrix} a_1 & 0 & 0 \\
0 & a_1 & 0 \\
0 & 0 & a_2 \end{bmatrix} \\
B_u(\phi) &= \begin{bmatrix} 2h_1 \cos(\phi) & b_1 \gamma_1 & b_2 \gamma_2 \\
2h_2 \sin(\phi) & b_3 \gamma_1 & b_4 \gamma_4 \\
b_2 & b_3 & b_4 \end{bmatrix} \\
\gamma_1 &= -\cos(\phi) - \sqrt{3} \sin(\phi), \\
\gamma_2 &= -\cos(\phi) + \sqrt{3} \sin(\phi), \\
\gamma_3 &= -\sin(\phi) + \sqrt{3} \cos(\phi), \\
\gamma_4 &= -\sin(\phi) - \sqrt{3} \cos(\phi)
\end{align*}$$
3. Trajectory Controller Design

The control objective of this paper is to design a suitable control law so that the position of the omnidirectional mobile robot can be kept at the desired position and allow the tracking error \( e = [e_x, e_y, e_z]^T \) to be zero. The nonlinear sliding mode control scheme is chosen to cope with the modeling uncertainty. The overall control framework of the sliding mode structure is illustrated in Fig. 5. For the present case, we assume that there are sensors such as wheel encoders to detect movement of the mobile robot. In order to reflect actual hardware limitation, saturation effect on the output voltage of motor actuators is also considered.

\[ U_c = B_u(\phi)^{-1}[\ddot{P}_d - A_u \dot{P}_w - U] \]  

Substituting (27) back into (21), the following linear model can be resulted

\[ \ddot{P}_w = \ddot{P}_d - U \]

3.2 The sliding mode controller

From the dynamic equation (22), the system can be formulated as follows:

\[ \ddot{P}_w + B_u(\phi)U_c + D_f \]

In order to have the system track \( P_m(t) = P_d(t) \), we define a sliding surface \( s = 0 \) expressed in terms of the error signals

\[
\begin{align*}
S_x &= (\dot{x}_w - \dot{x}_d) + k_x (x_w - x_d) = \dot{e}_x + k_x e_x \\
S_y &= (\dot{y}_w - \dot{y}_d) + k_y (y_w - y_d) = \dot{e}_y + k_y e_y \\
S_\phi &= (\dot{\phi}_w - \dot{\phi}_d) + k_\phi (\phi_w - \phi_d) = \dot{e}_\phi + k_\phi e_\phi 
\end{align*}
\]

where \( k_x, k_y, \) and \( k_\phi \) are positive constant parameters. Let the sliding surfaces and the friction function be represented in vector forms, \( S = [S_x \ S_y \ S_\phi]^T \) and \( D_f = [D_{fx} \ D_{fy} \ D_{\phi}]^T \), respectively. The error dynamics can be formulated as follows:

\[
\begin{bmatrix}
\dot{S}_x \\
\dot{S}_y \\
\dot{S}_\phi \\
\end{bmatrix} = -\begin{bmatrix}
\ddot{x}_d \\
\ddot{y}_d \\
\ddot{\phi}_d \\
\end{bmatrix} + \begin{bmatrix}
k_x & k_y & k_\phi \\
\end{bmatrix} \begin{bmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_\phi \\
\end{bmatrix} + \begin{bmatrix}
(A_u + \Delta A_u) & S_x \\
& S_y \\
& S_\phi \\
\end{bmatrix} \\
+ \begin{bmatrix}
B_u(\phi)E_1 \\
B_u(\phi)E_2 \\
B_u(\phi)E_3 \\
\end{bmatrix} + \begin{bmatrix}
D_{fs} \\
D_{fy} \\
D_{f\phi} \\
\end{bmatrix}
\]

Let the controller be chosen as:

\[
\begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
\end{bmatrix} = -B_u^{-1}(\phi)A_u \begin{bmatrix}
\dot{x}_w \\
\dot{y}_w \\
\dot{\phi}_w \\
\end{bmatrix} + B_u^{-1}(\phi) \begin{bmatrix}
\ddot{x}_d \\
\ddot{y}_d \\
\ddot{\phi}_d \\
\end{bmatrix} + \begin{bmatrix}
k_x \dot{e}_x \\
-k_\phi \dot{e}_\phi \\
\end{bmatrix} \begin{bmatrix}
k_x S_x \\
-k_\phi S_\phi \\
\end{bmatrix} + \begin{bmatrix}
-\eta_x \text{sign}(S_x) \\
-\eta_\phi \text{sign}(S_\phi) \\
\end{bmatrix}
\]

and \( k_x, k_\phi, \) and \( \eta \) are positive constant parameters, and \( \text{sign}(\cdot) \) is the sign function. If the sliding surfaces are
asymptotically stable then $e(t)$ converges asymptotically to zeros. In the following, it is shown that the error dynamics in (29), which is derived from the control law (30), is asymptotically stable in the sense of Lyapunov theory.

Define a Lyapunov candidate function to be $V = \frac{1}{2}S^TS$ and take its derivative with respect to time. Then, from (28) and (30), $\dot{V}$ can be derived as:

$$\dot{V} = S \dot{S} = S \Delta A_{\phi_\infty} \dot{x}_m + S \Delta A_{\phi_\infty} \dot{y}_m + S \Delta A_{\phi_\infty} \dot{\phi}_m - \eta_\phi [S_x] - \eta_\phi [S_y]$$

$$- \eta_\phi [S_x] + \left( -k_x S_x^2 - b_x S_y^2 - k_{\phi} S_{\phi}^2 \right) + S_x D_{x_\phi} + S_y D_{y_\phi} + S_\phi D_{\phi_\phi} \leq 0$$

$\dot{V}(t)$ is assured to be negative semi-definite, whenever choose $\eta_\phi > \max(\|\Delta A_{\phi_\infty}\|, D_{\phi_\phi})$, $\eta_\phi > \max(\|\Delta A_{\phi_\infty}\|, D_{\phi_\phi})$, $\eta_\phi > \max(\|\Delta A_{\phi_\infty}\|, D_{\phi_\phi})$, and $k_x$, $k_y$, and $k_\phi$ are positive definite. Moreover, the tracking error $e(t)$ will converge to zero according to $S(t) \rightarrow 0$ as $t \rightarrow \infty$ asymptotically and therefore, $e(t), \dot{e}(t) \rightarrow 0$ as $t \rightarrow \infty$. The effectiveness of the proposed omnidirectional mobile robot system will be verified by the following numerical simulation examples.

4. Simulation Results

In the section, the proposed sliding mode control approach will be applied to the tracking problem of the omnidirectional mobile robot. The omnidirectional mobile robot can independently achieve the translational motion and the rotational motion on the two dimensional plane. It was assumed that the robot’s initial position was $[0 \ 0 \ 0]$ for all tests, and the robot was stationary in the initial position. All physical parameters of the model and the default parameters are as follows:

$I_z = 0.2315 \text{kgm}^2 \quad R = 0.05 \text{m} \quad M = 10.5 \text{kg} \quad L = 0.18 \text{m} \quad \sigma = 28.2 \text{N}\cdot\text{m/rad} \quad F_c = 3 \text{N} \cdot \text{m}$

4.1 Without friction and model uncertainties

<table>
<thead>
<tr>
<th>Controller parameter</th>
<th>PID</th>
<th>Sliding mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[k_p \ k_i \ k_d]$</td>
<td>$[5 \ 2 \ 6]$</td>
<td>$[25 \ 25 \ 58]$</td>
</tr>
<tr>
<td>$[k_x \ k_y \ k_{\phi}]$</td>
<td>$[15 \ 15 \ 28]$</td>
<td>$[3 \ 3 \ 3]$</td>
</tr>
</tbody>
</table>

In the first case, the friction function and the model uncertainties are not considered. The tracking results including the tracking response, tracking error, and robot trajectory are shown in Fig. 6. The controller parameters are given as in Table 1 and the reference trajectory in the test is a circle defined as follows:

$x_d = 0.3\cos(2\pi t) - 0.3 \text{ m}$

$y_d = 0.3\sin(2\pi t) \text{ m}$

$\phi_d = -0.2\pi t$
In Fig. 6, the system tracking trajectory finally converged to the reference circle and the error states reduced to zero by both PID and sliding mode controllers. In this case, both the friction function and the model uncertainties are considered, and the tracking results are depicted in Fig. 7. The reference trajectory and controller parameters are the same as used in Section 4.1.

Fig. 7 apparently shows that the system tracking trajectory and the error states did not respectively converge to the reference circle and zero by the conventional PID controller. Nevertheless, by the sliding mode control algorithm, the system tracking trajectory was able to follow the reference circle, and error states finally converged to zeros.

4.2 With friction and model uncertainties

In this case, both the friction function and the model uncertainties are considered, and the tracking results are

5. Conclusion

In this paper, a robust sliding mode controller is developed to achieve perfect tracking for an omnidirectional mobile robot in the presence of parametric uncertainties and external disturbances. The nonlinear friction including slipping and traction forces is considered to reflect actual dynamic behavior of the mobile robot in the real world. According to the simulation results, the proposed controller can effectively overcome possible uncertainties and external disturbances, and accomplish excellent tracking tasks.

References

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