An extragradient-like approximation method for variational inequality problems and fixed point problems

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Abstract

The purpose of this paper is to investigate the problem of finding a common element of the set of fixed points of a non-expansive mapping and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. We introduce an extragradient-like approximation method which is based on so-called extragradient method and viscosity approximation method. We establish a strong convergence theorem for two iterative sequences generated by this method.

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Keywords: Extragradient-like approximation method; Variational inequality; Fixed point; Monotone mapping; Nonexpansive mapping; Strong convergence

1. Introduction

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $C$ be a nonempty closed convex subset of $H$. A mapping $A$ of $C$ into $H$ is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \forall x, y \in C.$$ 

$A$ is called $\beta$-inverse-strongly monotone (see [1,3]) if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \beta \| Ax - Ay \|^2 \quad \forall x, y \in C.$$ 

It is clear that a $\beta$-inverse-strongly monotone mapping $A$ is monotone and Lipschitz continuous.
In this paper, we consider the following variational inequality problem (VI($A, C$)): find a $\bar{x} \in C$ such that
$$\langle A\bar{x}, y - \bar{x} \rangle \geq 0 \quad \forall y \in C.$$  

The set of solutions of the VI($A, C$) is denoted by $\Omega$. A mapping $S : C \to C$ is called nonexpansive (see [7]) if
$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.$$  

The set of fixed points of $S$ is denoted by $F(S)$. Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\alpha > 0$ such that
$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \quad \forall x, y \in C.$$  

In addition, let $P_C : H \to C$ be the metric projection of $H$ onto $C$.

For finding an element of $F(S) \cap \Omega$ under the assumption that a set $C \subset H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is $\beta$-inverse-strongly monotone, Takahashi and Toyoda [8] introduced the following iterative scheme:

$$x_{n+1} = x_n + (1 - \lambda_n)SP_C(x_n - \lambda_nAx_n) \quad \forall n \geq 0,$$  \hspace{1cm}(1)  

where $x_0 = x \in C$, $\{x_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap \Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1) converges weakly to some $z \in F(S) \cap \Omega$. Recently, motivated by the idea of Korpelevich’s extragradient method [2], Nadezhkina and Takahashi [10] introduced an iterative scheme for finding an element of $F(S) \cap \Omega$ and presented the following weak convergence result.

**Theorem 1.1** [10, Theorem 3.1]. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $A : C \to H$ be a monotone, $k$-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by

$$\begin{cases}  
x_0 = x \in H, \\
y_n = P_C(x_n - \lambda_n Ax_n), \\
x_{n+1} = x_n + (1 - \lambda_n)SP_C(x_n - \lambda_n Ay_n) \quad \forall n \geq 0,
\end{cases}$$  \hspace{1cm}(2)  

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{x_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega$ where $z = \lim_{n \to \infty} P_{F(S) \cap \Omega} x_n$.

Very recently, inspired by Nadezhkina and Takahashi’s iterative scheme [10], Zeng and Yao [11] introduced another iterative scheme for finding an element of $F(S) \cap \Omega$ and obtained the following weak convergence theorem.

**Theorem 1.2** [11, Theorem 3.1]. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $A : C \to H$ be a monotone, $k$-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by

$$\begin{cases}  
x_0 = x \in H, \\
y_n = P_C(x_n - \lambda_n Ax_n), \\
x_{n+1} = x_n + (1 - \lambda_n)SP_C(x_n - \lambda_n Ay_n) \quad \forall n \geq 0,
\end{cases}$$  \hspace{1cm}(3)  

where $\{\lambda_n\}$ and $\{x_n\}$ satisfy the conditions:

(a) $\{\lambda_n\} \subset (0, 1 - \delta)$ for some $\delta \in (0, 1);$

(b) $\{x_n\} \subset (0, 1), \sum_{n=0}^{\infty} x_n = \infty, \lim_{n \to \infty} x_n = 0.$

Then the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $P_{F(S) \cap \Omega}(x_0)$ provided $\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0$.

On the other hand, in 2004, Xu [12] also considered so-called viscosity approximation method for finding a fixed point of a nonexpansive self-mapping on $C$ which solves some variational inequality. In this paper,
we introduce an extragradient-like approximation method which is based on the above extragradient method and viscosity approximation method, i.e.,

\[
\begin{cases}
x_0 = x \in C, \\
y_n = (1 - \gamma_n)x_n + \gamma_nP_C(x_n - \lambda_nAx_n), \\
x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_nf(y_n) + \beta_nSP_C(x_n - \lambda_nAy_n) \quad \forall n \geq 0,
\end{cases}
\]

where \(\{\lambda_n\}\) is a sequence in \((0, 1)\) with \(\sum_{n=0}^{\infty} \lambda_n < \infty\), and \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) are three sequences in \([0, 1]\) satisfying the conditions:

(i) \(\alpha_n + \beta_n \leq 1\) for all \(n \geq 0\);
(ii) \(\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty\);
(iii) \(0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1\).

It is shown that the sequence \(\{x_n\}, \{y_n\}\) generated by the above method converge strongly to the same point 
\(q = P_{\mathcal{F}(S)}xf(q)\) if and only if \(\{Ax_n\}\) is bounded and \(\liminf_{n \to \infty} \langle Ax_n, y - x_n \rangle \geq 0\) for all \(y \in C\).

2. Preliminaries

Let \(H\) be a real Hilbert space with inner product \(\langle \cdot, \cdot \rangle\) and norm \(\| \cdot \|\). It is well known that for all \(x, y \in H\) and \(\lambda \in [0, 1]\) there holds

\[
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.
\]

Let \(C\) be a nonempty closed convex subset of \(H\). Then for any \(x \in H\), there exists a unique nearest point \(u \in C\) such that

\[
\|x - u\| \leq \|x - y\| \quad \forall y \in C.
\]

The mapping \(P_C : x \to u\) is called the metric projection of \(H\) onto \(C\). It is known that \(P_C\) is nonexpansive. It is also known that \(P_C\) is characterized by the following properties (see \([7]\) for more details): \(P_Cx \in C\) and for all \(x \in H, y \in C\),

\[
\langle x - P_Cx, P_Cx - y \rangle \geq 0,
\]

and

\[
\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2.
\]

Let \(A : C \to H\) be a mapping. It is easy to see from (4) that the following implications hold:

\[
\tilde{x} \in \Omega \iff \tilde{x} = P_C(\tilde{x} - \lambda Ax) \quad \forall \lambda > 0.
\]

A set-valued mapping \(T : H \to 2^H\) is called monotone if for all \(x, y \in H, f \in Tx\) and \(g \in Ty\), we have \(\langle x - y, f - g \rangle \geq 0\). A monotone mapping \(T : H \to 2^H\) is maximal if its graph \(G(T)\) is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping \(T\) is maximal if and only if for \((x, f) \in H \times H, (x - y, f - g) \geq 0\) for all \((y, g) \in G(T)\), then \(f \in Tx\). Let \(A : C \to H\) be a monotone, \(L\)-Lipschitz continuous mapping and \(N_Cv\) be the normal cone to \(C\) at \(v \in C\), i.e., \(N_Cv = \{w \in H : \langle v - y, w \rangle \geq 0 \quad \forall y \in C\}\). Define

\[
Tv = \begin{cases}
Av + N_Cv, & \text{if } v \in C, \\
\emptyset, & \text{if } v \notin C.
\end{cases}
\]

Then \(T\) is maximal monotone and \(0 \in Tv\) if and only if \(v \in \Omega\); see \([5]\).
In order to prove the main result in Section 3, we shall use the following lemmas in the sequel.

**Lemma 2.1** [6, Lemma 2.5]. Let \( \{s_n\} \) be a sequence of nonnegative real numbers satisfying

\[
s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \beta_n + \gamma_n, \quad \forall n \geq 0,
\]

where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) satisfy the conditions:

1. \( \{\alpha_n\} \subset [0, 1], \sum_{n=0}^{\infty} \alpha_n = \infty, \) or equivalently, \( \prod_{n=0}^{\infty} (1 - \alpha_n) = 0; \)
2. \( \lim \sup_{n \to \infty} \beta_n \leq 0; \)
3. \( \gamma_n \geq 0 \) \( (n \geq 0), \sum_{n=0}^{\infty} \gamma_n < \infty. \)

Then \( \lim_{n \to \infty} s_n = 0. \)

**Lemma 2.2** ([4] Demiclosedness principle). Assume that \( S \) is a nonexpansive self-mapping of a nonempty closed convex subset \( C \) of a real Hilbert space \( H. \) If \( F(S) \neq \emptyset, \) then \( I - S \) is demiclosed; that is, whenever \( \{x_n\} \) is a sequence in \( C \) weakly converging to some \( x \in C \) and the sequence \( \{(I - S)x_n\} \) strongly converges to some \( y, \) it follows that \((I - S)x = y.\) Here \( I \) is the identity operator of \( H.\) The following lemma is an immediate consequence of an inner product.

**Lemma 2.3.** In a real Hilbert space \( H, \) there holds the inequality

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \quad \forall x, y \in H.
\]

**Lemma 2.4** [9]. Let \( \{x_n\} \) and \( \{z_n\} \) be bounded sequences in a Banach space \( X \) and let \( \{q_n\} \) be a sequence in \([0, 1]\) with \( 0 < \lim \inf_{n \to \infty} q_n \leq \lim \sup_{n \to \infty} q_n < 1.\) Suppose that \( x_{n+1} = q_n x_n + (1 - q_n)z_n \) for all integers \( n \geq 0 \) and \( \lim \sup_{n \to \infty} (\|x_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \) Then, \( \lim_{n \to \infty} (\|x_n\| - \|x_n - x_n\|) = 0. \)

Throughout this paper, we shall use the notations: “\( \rightharpoonup \) and “\( \to \)” to stand for the weak convergence and strong convergence, respectively.

### 3. Main result

Now we are in a position to state and prove the main result in this paper.

**Theorem 3.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H. \) Let \( f : C \to C \) be a contractive mapping with a contractive constant \( \lambda \in (0, 1), \) \( A : C \to H \) be a monotone, \( L\)-Lipschitz continuous mapping and \( S : C \to C \) be a nonexpansive mapping such that \( F(S) \cap \Omega \neq \emptyset. \) Let \( \{x_n\}, \{y_n\} \) be the sequences generated by

\[
\begin{cases}
    x_0 = x \in C, \\
    y_n = (1 - \gamma_n)x_n + \gamma_n PC(x_n - \lambda_n Ax_n), \\
    x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n f(y_n) + \beta_n SP_C(x_n - \lambda_n Ay_n),
\end{cases}
\]

where \( \{\lambda_n\} \) is a sequence in \((0, 1)\) with \( \sum_{n=0}^{\infty} \lambda_n < \infty, \) and \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are three sequences in \([0, 1]\) satisfying the conditions:

1. \( \alpha_n + \beta_n \leq 1 \) for all \( n \geq 0; \)
2. \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty; \)
3. \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1. \)

Then the sequences \( \{x_n\}, \{y_n\} \) converge strongly to the same point \( q = P_{F(S) \cap \Omega} f(q) \) if and only if \( \{Ax_n\} \) is bounded and \( \lim \inf_{n \to \infty} \langle Ax_n, y - x_n \rangle \geq 0 \) for all \( y \in C. \)

**Proof.** “Necessity”. Suppose that both \( \{x_n\} \) and \( \{y_n\} \) converge strongly to \( q = P_{F(S) \cap \Omega} f(q). \) Then from the Lipschitz continuity of \( A \) it follows that \( \{Ax_n\} \) is bounded, and for each \( y \in C \)
which implies that

$$\lim_{n \to \infty} \langle Ax_n, y - x_n \rangle = \langle Aq, y - q \rangle \geq 0 \quad \forall y \in C$$

due to $q \in \Omega$.

"Sufficiency." Suppose that $\{Ax_n\}$ is bounded and $\lim \inf_{n \to \infty} \langle Ax_n, y - x_n \rangle \geq 0$ for all $y \in C$. Then we divide the proof of the sufficiency into several steps.

Step 1. $\{x_n\}$ is bounded. Indeed, put $t_n = PC(x_n - \lambda_n Ay_n)$ for all $n \geq 0$. Let $x^* \in F(S) \cap \Omega$. Then $x^* = PC(x^* - \lambda_n Ax^*)$. Taking $x = x_n - \lambda_n Ay_n$ and $y = x^*$ in (5), we obtain

$$\begin{align*}
\|t_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - t_n\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda_n \langle Ay_n, x_n - x^*\rangle + \lambda_n^2 \|Ay_n\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle Ay_n, x_n - t_n\rangle - \lambda_n^2 \|Ay_n\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda_n \langle Ay_n, x^* - t_n\rangle - \|x_n - t_n\|^2 \\
&= \|x_n - x^*\|^2 - \|x_n - t_n\|^2 - 2\lambda_n \langle Ay_n - Ax^*, y_n - x^*\rangle + 2\lambda_n \langle Ay_n, x_n - t_n\rangle.
\end{align*}$$

(8)

Since $A$ is monotone and $x^*$ is a solution of the variational inequality problem VI($A, C$), we have

$$\langle xAy_n - Ax^*, y_n - x^*\rangle \geq 0 \quad \text{and} \quad \langle Ax^*, y_n - x^*\rangle \geq 0.$$

It follows from (8) that

$$\begin{align*}
\|t_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - x^*\|^2 - \|(x_n - y_n) + (y_n - t_n)\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle + \|y_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.
\end{align*}$$

(9)

Note that $x_n \in C$ for all $n \geq 0$ and that $y_n = (1 - \gamma_n)x_n + \gamma_n PC(x_n - \lambda_n Ax_n)$. Hence we have

$$\begin{align*}
2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle &\leq 2\|x_n - x^*\|^2 - \|t_n - y_n\|^2 + \lambda_n^2 \|Ay_n\|^2 + 2\langle x_n - y_n, \lambda_n Ay_n - y_n\rangle \\
&= \|x_n - y_n\|^2 + \|t_n - y_n\|^2 + 2\lambda_n \langle Ay_n, x_n - \lambda_n Ax_n - PCx_n\rangle + \lambda_n^2 \|Ay_n\|^2 \\
&\leq \|x_n - y_n\|^2 + \|t_n - y_n\|^2 + 2\lambda_n \|Ay_n\|\|PC(x_n - \lambda_n Ax_n) - PCx_n\| + \lambda_n^2 \|Ay_n\|^2 \\
&\leq \|x_n - y_n\|^2 + \|t_n - y_n\|^2 + 2\lambda_n \|Ay_n\|\|Ax_n - PCx_n\| + \lambda_n^2 \|Ay_n\|^2.
\end{align*}$$

(10)

Since $\{Ax_n\}$ is bounded and $A$ is $L$-Lipschitz continuous, we have

$$\|Ay_n - Ax_n\| \leq L\|y_n - x_n\| = L\|PC(x_n - \lambda_n Ax_n) - PCx_n\| \leq L\|Ax_n\|,$$

and hence $\|Ay_n\| \leq (1 + L)\|Ax_n\|$, which implies that $\{Ay_n\}$ is bounded. Put $M = \sup\{\|Ax_n\| + \|Ay_n\| : n \geq 0\}$. Then it follows from (10) that

$$\begin{align*}
2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle &\leq \|x_n - y_n\|^2 + \|t_n - y_n\|^2 + \lambda_n^2 (\|Ax_n\| + \|Ay_n\|)^2 \\
&\leq \|x_n - y_n\|^2 + \|t_n - y_n\|^2 + \lambda_n^2 M^2.
\end{align*}$$
This together with (9) implies that
\[
\|t_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n \rangle \\
\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \|x_n - y_n\|^2 + \|t_n - y_n\|^2 + \lambda_n^2 M^2 = \|x_n - x^*\|^2 + \lambda_n^2 M^2,
\]
and hence
\[
\|t_n - x^*\| \leq \|x_n - x^*\| + \lambda_n M. \tag{11}
\]
Observe that
\[
\|f(y_n) - x^*\| \leq \|f(y_n) - f(x^*)\| + \|f(x^*) - x^*\| \leq \lambda_n \|y_n - x^*\| + \|f(x^*) - x^*\| \\
= \lambda_n \|1 - \gamma_n\| (x_n - x^*) + \gamma_n (P_C (x_n - \lambda_n Ax_n) - P_C (x^* - \lambda_n Ax^*)) + \|f(x^*) - x^*\| \\
\leq \lambda_n \|1 - \gamma_n\| \|x_n - x^*\| + \gamma_n \| (x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\| + \|f(x^*) - x^*\| \\
\leq \lambda_n \|x_n - x^*\| + \gamma_n \| (\|x_n - x^*\| + \lambda_n \|Ax_n - Ax^*\|) + \|f(x^*) - x^*\| \\
\leq \lambda_n \|x_n - x^*\| + \lambda_n (M + \|Ax^*\|) + \|f(x^*) - x^*\|. \\
\]
which together with (11) implies that
\[
\|x_{n+1} - x^*\| = \|(1 - \lambda_n - \beta_n)x_n + \lambda_n f(y_n) + \beta_n St_n - x^*\| \\
\leq \|(1 - \lambda_n - \beta_n)x_n - x^*\| + \lambda_n \|f(y_n) - x^*\| + \beta_n \|St_n - x^*\| \\
\leq \|(1 - \lambda_n - \beta_n)x_n - x^*\| + \lambda_n \|x_n - x^*\| + \lambda_n \|M + \|Ax^*\|\| + \|f(x^*) - x^*\| \\
+ \beta_n \|x_n - x^*\| + \lambda_n \lambda_n (M + \|Ax^*\|) \\
= (1 - (1 - \lambda_n) \lambda_n \|x_n - x^*\| + \lambda_n \|f(x^*) - x^*\| + (\lambda_n + \beta_n) \lambda_n M + \lambda_n \|Ax^*\| \\
\leq (1 - (1 - \lambda_n) \lambda_n \|x_n - x^*\| + \lambda_n \|f(x^*) - x^*\| + \lambda_n (M + \|Ax^*\|) \\
\leq (1 - \lambda_n) \lambda_n \|x_n - x^*\| + \lambda_n \|f(x^*) - x^*\| + \lambda_n (M + \|Ax^*\|). \\
\]
We shall prove that
\[
\|x_{n+1} - x^*\| \leq \max \left\{ \|x_0 - x^*\|, \frac{1}{1 - \lambda_n} \|f(x^*) - x^*\|, \sum_{j=0}^n \lambda_j (M + \|Ax^*\|) \right\} \tag{12}
\]
for all \(n \geq 0\). Whenever \(n = 0\), it is easy to see that
\[
\|x_1 - x^*\| \leq (1 - (1 - \lambda_n) \lambda_n) \|x_0 - x^*\| + (1 - \lambda_n) \lambda_n \cdot \frac{1}{1 - \lambda_n} \|f(x^*) - x^*\| + \lambda_0 (M + \|Ax^*\|) \\
\leq \max \left\{ \|x_0 - x^*\|, \frac{1}{1 - \lambda_n} \|f(x^*) - x^*\| \right\} + \lambda_0 (M + \|Ax^*\|). \\
\]
This shows that (12) holds. Now suppose that (12) holds for \(n \geq 0\). Then we have
\[
\|x_{n+1} - x^*\| \leq (1 - (1 - \lambda_n) \lambda_n) \|x_n - x^*\| + (1 - \lambda_n) \lambda_n \cdot \frac{1}{1 - \lambda_n} \|f(x^*) - x^*\| + \lambda_0 (M + \|Ax^*\|) \\
\leq (1 - (1 - \lambda_n) \lambda_n) \max \left\{ \|x_0 - x^*\|, \frac{1}{1 - \lambda_n} \|f(x^*) - x^*\| \right\} + \sum_{j=0}^{n-1} \lambda_j (M + \|Ax^*\|) \\
\cdot \frac{1}{1 - \lambda_n} \|f(x^*) - x^*\| + \lambda_0 (M + \|Ax^*\|) \\
\leq \max \left\{ \|x_0 - x^*\|, \frac{1}{1 - \lambda_n} \|f(x^*) - x^*\| \right\} + \sum_{j=0}^{n-1} \lambda_j (M + \|Ax^*\|) + \lambda_n (M + \|Ax^*\|) \\
= \max \left\{ \|x_0 - x^*\|, \frac{1}{1 - \lambda_n} \|f(x^*) - x^*\| \right\} + \sum_{j=0}^n \lambda_j (M + \|Ax^*\|). \\
\]
Hence (12) holds for $n + 1$. By induction we know that (12) holds for all $n \geq 0$. Consequently, $\{x_n\}$ is bounded and so are the sequences $\{t_n\}$, $\{S_n\}$ by (11).

Step 2. $\lim_{n \to \infty} ||x_{n+1} - x_n|| = 0$. Indeed, observe that

$$
\|x_{n+1} - x_n\| = \|P_C(x_{n+1} - x_n)\| \leq \|x_{n+1} - x_n\| + \|A_y_{n+1}\| + \|A_y_n\| \leq ||x_{n+1} - x_n|| + (\lambda_n + \lambda_{n+1})M,
$$

and

$$
\|y_{n+1} - y_n\| = \|(1 - \gamma_{n+1})x_{n+1} + \gamma_{n+1}P_C(x_{n+1} - x_n)\| + \|(1 - \gamma_n)x_n - \gamma_nP_C(x_n - x_n)\|
+ \|(\gamma_{n+1} - \gamma_n)P_C(x_n - x_{n+1})\| + \|(\gamma_{n+1} - \gamma_n)P_C(x_n - x_{n+1})\| + \|x_{n+1} - x_n\| + \|Ax_{n+1}\| + \|Ax_n\|
$$

$$
\leq ||x_{n+1} - x_n|| + \|x_{n+1} - x_n\| + \|Ax_{n+1}\| + \|Ax_n\| \leq ||x_{n+1} - x_n|| + (2\lambda_n + \lambda_{n+1})M. \tag{14}
$$

Define a sequence $\{z_n\}$ by

$$
x_{n+1} = q_nx_n + (1 - q_n)z_n, \quad n \geq 0,
$$

where $q_n = 1 - \alpha_n - \beta_n, n \geq 0$. Then we have

$$
z_{n+1} - z_n = \frac{x_{n+1} - q_nx_n}{1 - q_n} = \frac{x_{n+1} - q_nx_n}{1 - q_n} = \frac{\alpha_{n+1}f(y_{n+1}) + \beta_{n+1}St_{n+1}}{1 - q_n} - \frac{\alpha_nf(y_n) + \beta_nSt_n}{1 - q_n}
= \frac{\alpha_{n+1}}{1 - q_n} (f(y_{n+1}) - f(y_n)) + \frac{\alpha_{n+1}}{1 - q_n} (St_{n+1} - St_n) + \frac{\alpha_n}{1 - q_n - q_n} (St_{n+1} - St_n)
$$

$$
= \frac{\alpha_{n+1}}{1 - q_n} (f(y_{n+1}) - f(y_n)) + \frac{\alpha_{n+1}}{1 - q_n - q_n} (St_{n+1} - St_n).
$$

From (13)–(15), we get

$$
\|z_{n+1} - z_n\| \leq \frac{\alpha_{n+1}}{1 - q_n} \|f(y_{n+1}) - f(y_n)\| + \frac{\alpha_{n+1}}{1 - q_n} \|f(y_n)\| + \frac{\beta_{n+1}}{1 - q_n} \|St_{n+1} - St_n\|
+ \frac{\alpha_n}{1 - q_n - q_n} (||St_{n+1} - St_n|| + ||St_n||)
$$

$$
\leq \frac{\alpha_{n+1}}{1 - q_n} \|y_{n+1} - y_n\| + \frac{\alpha_{n+1}}{1 - q_n} \|f(y_n)\| + \frac{\alpha_n}{1 - q_n} (||f(y_n)|| + ||St_n||) + \frac{\beta_{n+1}}{1 - q_n} \|t_{n+1} - t_n\|
$$

$$
\leq \frac{\alpha_{n+1}}{1 - q_n} \|x_{n+1} - x_n\| + (2\lambda_n + \lambda_{n+1})M + \frac{\alpha_{n+1}}{1 - q_n} \|f(y_n)\| + ||St_n||
+ \frac{\beta_{n+1}}{1 - q_n} \|t_{n+1} - t_n\|
$$

$$
\leq ||x_{n+1} - x_n|| + (2\lambda_n + \lambda_{n+1})M + \frac{\alpha_{n+1}}{1 - q_n} \|f(y_n)\| + ||St_n||,
$$

which implies that

$$
\|z_{n+1} - z_n\| - ||x_{n+1} - x_n|| \leq (2\lambda_n + \lambda_{n+1})M + \frac{\alpha_{n+1}}{1 - q_n} \|f(y_n)\| + ||St_n||.
$$
Note that the boundedness of \( \{x_n\} \) implies that \( \{f(x_n)\} \) is also bounded. Since
\[
\|y_n - x_n\| = \gamma_n\|PC(x_n - \lambda_nAx_n) - PCx_n\| \leq \lambda_n\|Ax_n\| \leq \lambda_nM \to 0,
\] (17)
we know that \( \{y_n\} \) is bounded and so is \( \{f(y_n)\} \). Moreover, \( \{t_n\} \) is bounded by (11) and hence \( \{St_n\} \) is also bounded. Also, note that
\[
\limsup_{n \to \infty} \frac{x_n}{1 - q_n} = \limsup_{n \to \infty} \frac{x_n}{x_n + \beta_n} \leq \limsup_{n \to \infty} \frac{x_n}{\beta_n} = 0
\]
by conditions (ii) and (iii). Thus we deduce from (16) that
\[
\limsup_{n \to \infty} (|z_{n+1} - z_n| - |x_{n+1} - x_n|) = 0.
\]
Since \( q_n = 1 - x_n - \beta_n \), we know from conditions (ii) and (iii) that
\[
0 < \liminf_{n \to \infty} q_n \leq \limsup_{n \to \infty} q_n < 1.
\]
Thus in terms of Lemma 2.4 we get \( \lim_{n \to \infty} |z_n - x_n| = 0 \). Consequently,
\[
\lim_{n \to \infty} |x_{n+1} - x_n| = \lim (1 - q_n)|z_n - x_n| = 0.
\] (18)

**Step 3.** \( \lim_{n \to \infty} |Sx_n - x_n| = \lim_{n \to \infty} |St_n - t_n| = 0 \). Indeed, observe that
\[
\|y_n - t_n\| = \|(1 - \gamma_n)(PCx_n - PC(x_n - \lambda_nAy_n)) + \gamma_n(PC(x_n - \lambda_nAx_n) - PC(x_n - \lambda_nAy_n))\|
\]
\[
\leq (1 - \gamma_n)\|PCx_n - PC(x_n - \lambda_nAy_n)\| + \gamma_n\|PC(x_n - \lambda_nAx_n) - PC(x_n - \lambda_nAy_n)\|
\]
\[
\leq \lambda_n\|Ax_n + \lambda_n\|Ax_n - Ay_n\| \to 0,
\]
\[
|t_n - x_n| \leq |t_n - y_n| + |y_n - x_n| \to 0,
\]
and hence
\[
|Sx_n - x_{n+1}| \leq |Sx_n - St_n| + |St_n - x_{n+1}| \leq |y_n - t_n| + (1 - x_n - \beta_n)\|St_n - x_n\| + x_n\|St_n - f(y_n)\|
\]
\[
\leq |y_n - t_n| + x_n\|St_n - f(y_n)\| + (1 - x_n - \beta_n)\|St_n - Sx_n\| + \|Sx_n - x_n\|
\]
\[
\leq |y_n - t_n| + x_n\|St_n - f(y_n)\| + |t_n - x_n| + (1 - \beta_n)\|Sx_n - x_n\|.
\]
Thus from the last three inequalities we conclude that
\[
|Sx_n - x_n| \leq |Sx_n - Sx_n| + |Sx_n - x_{n+1}| + |x_{n+1} - x_n|
\]
\[
\leq |x_n - y_n| + |y_n - t_n| + x_n\|St_n - f(y_n)\| + |t_n - x_n| + (1 - \beta_n)\|Sx_n - x_n\| + |x_{n+1} - x_n|
\]
\[
\leq 2(|y_n - t_n| + |t_n - x_n|) + x_n\|St_n - f(y_n)\| + (1 - \beta_n)\|Sx_n - x_n\| + |x_{n+1} - x_n|.
\]
Since \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \), we deduce from the boundedness of \( \{St_n\} \) and \( \{f(y_n)\} \) that
\[
\|Sx_n - x_n\| \to 0.
\]
Consequently,
\[
|St_n - t_n| \leq |St_n - Sx_n| + |Sx_n - x_n| + |x_n - t_n| \leq 2|t_n - x_n| + |Sx_n - x_n| \to 0.
\]

**Step 4.** \( \limsup_{n \to \infty} (f(q) - q, x_n - q) \leq 0 \). Indeed, we pick a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) so that
\[
\limsup_{n \to \infty} (f(q) - q, x_n - q) = \lim_{i \to \infty} (f(q) - q, x_{n_i} - q).
\] (19)

Without loss of generality, let \( x_{n_i} \to \hat{x} \in C \). Then (19) reduces to
\[
\limsup_{n \to \infty} (f(q) - q, x_n - q) = (f(q) - q, \hat{x} - q).
\]
In order to show \( (f(q) - q, \hat{x} - q) \leq 0 \), it suffices to show that \( \hat{x} \in F(S) \cap \Omega \). Note that by Lemma 2.2 and Step 3, we have \( \hat{x} \in F(S) \). Now we claim that \( \hat{x} \in \Omega \). Let
\[
Tv = \begin{cases} 
Av + N_C v, & \text{if } v \in C, \\
\emptyset, & \text{if } v \not\in C.
\end{cases}
\]

Then \( T \) is maximal monotone and \( 0 \in Tv \) if and only if \( v \in \Omega \); see [5]. Let \( (v, w) \in G(T) \). Then we have \( w \in Tv = Av + N_C v \) and hence \( w - Av \in N_C v \). Therefore we have \( \langle v - u, w - Av \rangle \geq 0 \) for all \( u \in C \). In particular, taking \( u = x_n, \) we get
\[
\langle v - \hat{x}, w \rangle = \liminf_{t \to \infty} \langle v - x_n, w \rangle \geq \liminf_{t \to \infty} \langle v - x_n, Av \rangle = \liminf_{t \to \infty} \left[ \langle v - x_n, Av - Ax_n \rangle + \langle v - x_n, Ax_n \rangle \right] 
\]
\[
\geq \liminf_{t \to \infty} \langle v - x_n, Ax_n \rangle \geq \liminf_{t \to \infty} \langle v - x_n, Av \rangle \geq 0
\]
and so \( \langle v - \hat{x}, w \rangle \geq 0 \). Since \( T \) is maximal monotone, we have \( \hat{x} \in T^{-1}0 \) and hence \( \hat{x} \in \Omega \). This shows that \( \hat{x} \in F(S) \cap \Omega \). Therefore by the property of the metric projection, we derive \( \langle f(q) - q, \hat{x} - q \rangle \leq 0 \).

Step 5. \( \lim_{n \to \infty} \|x_n - q\| = 0 \) where \( q = P_{F(S) \cap \Omega} f(q) \). Indeed from (7), (11) and Lemma 2.3 we get
\[
\|x_{n+1} - q\|^2 = \| (1 - \alpha_n - \beta_n) (x_n - q) + \alpha_n (f(y_n) - q) + \beta_n (Ax_n - q) \|^2 
\]
\[
\leq \| (1 - \alpha_n - \beta_n) (x_n - q) + \beta_n (Ax_n - q) \|^2 + 2 \alpha_n \| f(y_n) - q, x_{n+1} - q \| 
\]
\[
\leq \| (1 - \alpha_n - \beta_n) (x_n - q) + \beta_n (Ax_n - q) \|^2 + 2 \alpha_n \| f(y_n) - q, x_{n+1} - q \| 
\]
\[
\leq \| (1 - \alpha_n - \beta_n) (x_n - q) + \beta_n (Ax_n - q) \|^2 + 2 \alpha_n \| f(y_n) - q, x_{n+1} - q \| 
\]
\[
= \| (1 - \alpha_n) (x_n - q) + \beta_n M \|^2 + 2 \alpha_n \| f(y_n) - f(x_n), x_{n+1} - q \| + \| f(x_n) - f(q), x_{n+1} - q \| 
\]
\[
+ \| f(q) - q, x_{n+1} - q \| 
\]
\[
\leq \| (1 - \alpha_n) (x_n - q) \|^2 + \beta_n M (2 \| x_n - q \| + \beta_n M) + 2 \alpha_n \| x_n - x_n \| \| x_{n+1} - q \| 
\]
\[
+ \| f(q) - q, x_{n+1} - q \| 
\]
\[
\leq \| (1 - \alpha_n) (x_n - q) \|^2 + \beta_n M (2 \| x_n - q \| + \beta_n M) + 2 \alpha_n \| x_n - x_n \| \| x_{n+1} - q \| 
\]
\[
+ \| f(q) - q, x_{n+1} - q \| 
\]
which implies that
\[
\|x_{n+1} - q\|^2 \leq \frac{(1 - \alpha_n)^2 + 2 \alpha_n \| x_n - x_n \| \| x_{n+1} - q \| + \| f(q) - q, x_{n+1} - q \|}{1 - 2(1 - \alpha_n) \alpha_n + 2 \alpha_n \| x_n - x_n \|} + \frac{1}{1 - 2(1 - \alpha_n) \alpha_n} \| x_n - q \| + \| f(q) - q, x_{n+1} - q \| 
\]
\[
+ \frac{1}{1 - 2(1 - \alpha_n) \alpha_n} \| x_n - q \| + \| f(q) - q, x_{n+1} - q \| 
\]
\[
= (1 - 2(1 - \alpha_n) \alpha_n) \| x_n - q \|^2 + 2(1 - \alpha_n) \alpha_n 
\]
\[
+ \frac{1}{1 - 2(1 - \alpha_n) \alpha_n} \| x_n - q \|^2 + \| f(q) - q, x_{n+1} - q \| 
\]
\[
= \frac{1}{(1 - \alpha_n)(1 - \alpha_n)} \left[ \| x_n - q \|^2 + \alpha \| x_n - x_n \| \| x_{n+1} - q \| + \| f(q) - q, x_{n+1} - q \| \right] 
\]
\[
+ \frac{1}{1 - 2(1 - \alpha_n) \alpha_n} \| x_n - q \| + \| f(q) - q, x_{n+1} - q \| 
\]
Note that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} 2(1 - \alpha_n) \alpha_n = \infty \). Since \( \limsup_{n \to \infty} \| f(q) - q, x_{n+1} - q \| \leq 0 \), \( \lim_{n \to \infty} \| x_n - x_n \| = 0 \) and \( \{ x_n - q \} \) is bounded, we know that
\[
\limsup_{n \to \infty} \frac{1}{(1 - \alpha_n)(1 - \alpha_n)} \left[ \| x_n - q \|^2 + \alpha \| x_n - x_n \| \| x_{n+1} - q \| + \| f(q) - q, x_{n+1} - q \| \right] \leq 0.
\]
Also, it is easy to see that
\[
\sum_{n=0}^{\infty} \frac{1}{1 - 2(1 - \alpha_n) \alpha_n} \| x_n - q \| + \| f(q) - q, x_{n+1} - q \| < \infty.
\]
4. Applications

Next we give two applications of Theorem 3.1.

**Theorem 4.1.** Let $H$ be a real Hilbert space. Let $f : H \to H$ be a contractive mapping with constant $\alpha \in (0, 1)$, $A : H \to H$ be a monotone, $L$-Lipschitz continuous mapping and $S : H \to H$ be a nonexpansive mapping such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by

$$
\begin{align*}
  x_0 &= x \in H, \\
  y_n &= (1 - \gamma_n)x_n + \gamma_n(x_n - \lambda_nAx_n), \\
  x_{n+1} &= (1 - \beta_n)x_n + \alpha_n f(y_n) + \beta_n S(x_n - \lambda_nAy_n) \quad \forall n \geq 0,
\end{align*}
$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n + \gamma_n \leq 1$ for all $n \geq 0$;
(ii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

Then the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $q = P_{F(S) \cap A^{-1}0} f(q)$ if and only if $\{Ax_n\}$ is bounded and $\liminf_{n \to \infty} \langle Ax_n, y - x_n \rangle \geq 0$ for all $y \in H$.

**Proof.** We have $A^{-1}0 = \Omega$ and $P_H = I$ the identity mapping of $H$. By Theorem 3.1 we obtain the desired result. □

**Theorem 4.2.** Let $H$ be a real Hilbert space. Let $f : H \to H$ be a contractive mapping with constant $\alpha \in (0, 1)$, $A : H \to H$ be a monotone, $L$-Lipschitz continuous mapping and $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let $J^B_r$ be the resolvent of $B$ for each $r > 0$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by

$$
\begin{align*}
  x_0 &= x \in H, \\
  y_n &= (1 - \gamma_n)x_n + \gamma_n(x_n - \lambda_nAx_n), \\
  x_{n+1} &= (1 - \beta_n)x_n + \alpha_n f(y_n) + \beta_n J^B_r(x_n - \lambda_nAy_n) \quad \forall n \geq 0,
\end{align*}
$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n + \gamma_n \leq 1$ for all $n \geq 0$;
(ii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

Then the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $q = P_{A^{-1}0 \cap B^{-1}0} f(q)$ if and only if $\{Ax_n\}$ is bounded and $\liminf_{n \to \infty} \langle Ax_n, y - x_n \rangle \geq 0$ for all $y \in C$.

**Proof.** We have $A^{-1}0 = \Omega$ and $F(J^B_r) = B^{-1}0$. Putting $P_H = I$, by Theorem 3.1 we obtain the desired result. □

References